PERGAMON Computers and Mathematics with Applications 39 (2000) 25–43

An International Journal
computers & mathematics with applications

www.elsevier.nl/locate/camwa

Detecting the Singularities of a Function of V_p Class by its Integrated Fourier Series

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(Received and accepted October 1999)

Abstract—In the present paper, we pursue the general idea suggested in our previous work. Namely, we utilize the truncated Fourier series as a tool for the approximation of the points of discontinuities and the magnitudes of jumps of a 2π-periodic bounded function. Earlier, we used the derivative of the partial sums, while in this work we use integrals.

First, we obtain new identities which determine the jumps of a 2π-periodic function of V_p, $1 \le p < 2$, class, with a finite number of discontinuities, by means of the tails of its integrated Fourier series.

Next, based on these identities we establish asymptotic expansions for the approximations of the location of the discontinuity and the magnitude of the jump of a 2π-periodic piecewise smooth function with one singularity. By an appropriate linear combination, obtained via integrals of different order, we significantly improve the accuracy of the initial approximations. Then, we apply Richardson's extrapolation method to enhance the approximation results. For a function with multiple discontinuities we use simple formulae which "eliminate" all discontinuities of the function but one. Then we treat the function as if it had one singularity.

Finally, we give the description of a programmable algorithm for the approximation of the discontinuities, investigate the stability of the method, study its complexity, and present some numerical results.

Keywords—Detecting singularities, Fourier series, Asymptotic expansions.

1. INTRODUCTION

Locating the discontinuities of a function by means of its truncated Fourier series, an interesting problem in and of itself, arises naturally from an attempt to overcome the Gibbs phenomenon: the oscillatory behavior of the Fourier partial sums of a discontinuous function.

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The first author was supported by Research, Scholarship, and Professional Growth Grant, Weber State University, Ogden, UT. The second author was partially supported by National Science Foundation Grant DMS-9600146, and ICOMP, NASA-Glenn Research Center, Cleveland, OH.

PII: S0898-1221(00)00084-5

In [1], Cai, Gottlieb, and Shu developed the idea already introduced in [2–5], and suggested a method for the reconstruction of a discontinuous function from the partial sums of its Fourier series. A key step of the method is the accurate approximation of the location of singularities and the magnitudes of jumps of the function. Namely, let f be a 2π-periodic function which is piecewise smooth on the period with a finite number, M, of jump discontinuities. In addition, let us assume that the first $2n+1$ Fourier coefficients of the function are known. If $G(x) = (\pi - x)/2$, $x \in (0, 2\pi)$, is the 2π-periodic sawtooth function, then the function f can be represented as follows:

$$f(x) = \frac{1}{\pi} \sum_{m=0}^{M-1} [f]_m G(x - x_m) + f_c(x), \tag{1}$$

where x_m and $[f]_m$, $m = 0, 1, \dots, M-1$, are the locations of discontinuities and the associated jumps of the function f, and f_c is a 2π-periodic continuous function, which is piecewise smooth on $[-\pi, \pi]$.

Thus, the problem is to find a good approximation to the constants x_m and $[f]_m$, given the first $2n+1$ Fourier coefficients of the function f. Then f_c could be recovered from the partial sums of its Fourier series using identity (1) and the undesirable Gibbs phenomenon could be avoided.

A number of authors have proposed techniques for locating the discontinuities and determining the jumps [6–12]. These are discussed in more details in [13], where we propose an essentially different approach, based on derivatives of the Fourier series. Here we consider a closely related alternative based on integration rather than differentiation.

We begin by establishing special formulae which determine the jumps of a 2π-periodic function of V_p, $1 \le p < 2$, class, with a finite number of jump discontinuities, by means of the tails of its integrated Fourier series. Then, we utilize these formulae as a tool for the approximation of the discontinuities and the jumps of the function. Next, based on the identities we obtain asymptotic expansions for the approximations of the location of the discontinuity and the magnitude of the jump of a 2π-periodic piecewise smooth function with one singularity. By an appropriate linear combination, obtained via integrals of different order, we significantly improve the accuracy of the initial approximations. Then, combining the expansion formula with Richardson's extrapolation method, we further refine the accuracy. For a function with multiple discontinuities we use simple formulae which "eliminate" all discontinuities of the function but one. Then we treat the function as if it had one discontinuity following the method described above.

Finally, we give the description of a programmable algorithm for the approximation of the discontinuities, investigate the stability of the method, study its complexity, and consider some numerical examples.

Our results with integration formulae are quite similar to those obtained in [13]. We obtain slightly less accuracy than Eckhoff or Geer and Banerjee. However, unlike them, we automatically treat arbitrary numbers of discontinuities (as can Bauer, though with less accuracy than we obtain). Moreover, our theory is complete.

In the context of Fourier series, differentiation is quite simple, so that the need for an integration-based alternative is unclear. However, anticipating the generalization of these methods to Jacobi series, we recall in that context differentiation operators are significantly less sparse than integration operators, so that techniques based on the latter will probably be more efficient. Moreover, from the point of view of harmonic analysis, the formulas developed here are new, and perhaps, somewhat surprising. The differentiation formulas, on the other hand, have been known for some time.

2. DEFINITIONS AND LEMMAS

Throughout this paper, we use the following general notations: N, Z_+, Z, and R are the sets of positive integers, nonnegative integers, integers, and real numbers, respectively.

By $C^q[a,b]$, $q \in Z_+$, we denote the space of q-times continuously differentiable functions on $[a,b]$, where $C^0[a,b] \equiv C[a,b]$ is the space of continuous functions. By $C^{-1}[a,b]$ we denote the space of functions defined on $[a,b]$ which may have discontinuities only of the first kind and which are normalized by the condition $f(x) = (f(x+) + f(x-))/2$. (Here, and elsewhere, $f(x+)$ and $f(x-)$ mean the right- and left-hand side limits of a function f at a point x, respectively.)

DEFINITION. *(See [14].) A function f is said to have p-bounded variation on $[a,b]$, i.e., $f \in V_p[a,b]$, $p \geq 1$, if*

$$\sup_{\Pi} \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})|^p < \infty,$$

where $\Pi = \{a \leq t_0 < t_1 < \cdots < t_n \leq b\}$ is an arbitrary partition of $[a,b]$.

It is obvious that if $p = 1$, $V_p[a,b]$ coincides with the Jordan class $V[a,b]$ of functions of bounded variation. It is known that V_p is a linear space and $V_p \subset V_q$ for $1 \leq p \leq q$ [14].

If there is no ambiguity, we shall usually omit the dependence on the domain and simply refer to one of the introduced classes of functions as C^{-1}, V, etc.

By $\mathcal{A}, \mathcal{B}, \dots$, we denote constants, possibly depending on some fixed parameters and in general distinct in different formulae. For positive quantities $\mathcal{A}_n$ and $\mathcal{B}_n$, possibly depending on some other variables as well, we write $\mathcal{A}_n = o(\mathcal{B}_n)$, $\mathcal{A}_n = O(\mathcal{B}_n)$, or $\mathcal{A}_n \simeq \mathcal{B}_n$, if $\lim_{n\to\infty} \mathcal{A}_n/\mathcal{B}_n = 0$, $\sup_{n\in N} \mathcal{A}_n/\mathcal{B}_n < \infty$, or $\mathcal{K}_1 < \mathcal{A}_n/\mathcal{B}_n < \mathcal{K}_2$, respectively, where $\mathcal{K}_1 > 0$ and $\mathcal{K}_2 > 0$ are some absolute constants.

All functions below are assumed to be 2π-periodic with the obvious exceptions.

If a function f is integrable on $[-\pi,\pi]$, then it has a Fourier series with respect to the trigonometric system $\{1, \cos nx, \sin nx\}_{n=1}^{\infty}$, and we denote the n^{th} partial sum of the Fourier series of f by $S_n(f;\cdot)$, i.e.,

$$\begin{aligned} S_n(f;x) &= \frac{a_0(f)}{2} + \sum_{k=1}^{n} (a_k(f)\cos kx + b_k(f)\sin kx) \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(t-x)\,dt, \end{aligned} \tag{2}$$

where

$$a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt \qquad \text{and} \qquad b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt$$

are the k^{th} Fourier coefficients of the function f, and

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx = \begin{cases} \dfrac{\sin(n+(1/2))x}{2\sin(x/2)}, & \text{for } x \notin 2\pi Z, \\ n + \dfrac{1}{2}, & \text{for } x \in 2\pi Z, \end{cases} \tag{3}$$

is the Dirichlet kernel.

By $\tilde{S}_n(f;\cdot)$ we denote the n^{th} partial sum of the series conjugate to (2), i.e.,

$$\tilde{S}_n(f;x) = \sum_{k=1}^{n} (a_k(f)\sin kx - b_k(f)\cos kx) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\tilde{D}_n(t-x)\,dt,$$

where

$$\tilde{D}_n(x) = \sum_{k=1}^{n} \sin kx = \begin{cases} \dfrac{\cos(x/2) - \cos(n+(1/2))x}{2\sin(x/2)}, & \text{for } x \notin 2\pi Z, \\ 0, & \text{for } x \in 2\pi Z, \end{cases} \tag{4}$$

is the kernel conjugate to the Dirichlet kernel.

Correspondingly, by $R_n(f;\cdot)$ and $\tilde{R}_n(f;\cdot)$ we denote the n^{th} order tails of the Fourier and the conjugate to the Fourier series of the function f, i.e.,

$$R_n(f;x) = \sum_{k=n}^{\infty} (a_k(f)\cos kx + b_k(f)\sin kx)$$

and

$$\tilde{R}_n(f;x) = \sum_{k=n}^{\infty} (a_k(f)\sin kx - b_k(f)\cos kx),$$

for $n \in N$.

The r^{th} derivative of a function f, which piecewise belongs to C^q, $q \geq r$, or which belongs to C^{r-1}, we define as $f^{(r)}(x) = (f^{(r)}(x+) + f^{(r)}(x-))/2$, whenever $f^{(r)}(x\pm)$ exist. $f^{(-r)}$, $r \in Z_+$, is defined as follows: for any function f, integrable on $[-\pi,\pi]$,

$$f^{(-r-1)} \equiv \int f^{(-r)},$$

where $f^{(0)} \equiv f$, and the constants of integration are successively determined by the condition

$$\int_{-\pi}^{\pi} f^{(-r)}(t)\,dt = 0, \qquad r \in Z_+.$$

If $\beta \in R$ is fixed, then for the sawtooth function G we set $G(\beta;x) \equiv G(x-\beta)$.

Let us mention, as it trivially follows from (3) and (4), that

$$|D_{n,m}(\beta)|, \left|\tilde{D}_{n,m}(\beta)\right| < \frac{\mathcal{K}}{|\beta|} \tag{5}$$

for $0 < |\beta| \leq \pi$ and any $n \leq m$ $(n,m \in N)$, where $D_{n,m} \equiv D_{m-1} - D_{n-1}$ and $\tilde{D}_{n,m} \equiv \tilde{D}_{m-1} - \tilde{D}_{n-1}$.

Besides, it is straightforward to check that

$$\begin{aligned} R_{n,2n}^{(r+1)}\left(G^{(-q)}(\beta;\cdot);x\right) &= D_{n,2n}^{(r-q)}(x-\beta) \\ &= \sum_{k=n}^{2n-1} k^{r-q}\cos\left(k(x-\beta) + \frac{(r-q)\pi}{2}\right) \end{aligned} \tag{6}$$

and

$$\begin{aligned} \tilde{R}_{n,2n}^{(r+1)}\left(G^{(-q)}(\beta;\cdot);x\right) &= \tilde{D}_{n,2n}^{(r-q)}(x-\beta) \\ &= \sum_{k=n}^{2n-1} k^{r-q}\sin\left(k(x-\beta) + \frac{(r-q)\pi}{2}\right), \end{aligned} \tag{7}$$

for $r \in Z$, $q \in Z_+$, and $\beta \in R$, where $R_{n,m}(g;\cdot) \equiv R_n(g;\cdot) - R_m(g;\cdot)$, and $\tilde{R}_{n,m}(g;\cdot) \equiv \tilde{R}_n(g;\cdot) - \tilde{R}_m(g;\cdot)$, $n \leq m$.

By $M \equiv M(f)$ we denote the number of discontinuities of the function $f \in C^{-1}$. By $x_m \equiv x_m(f)$ and $[f]_m \equiv f(x_m+) - f(x_m-)$, $m = 0,1,\ldots,M-1$, we denote the points of discontinuity and the associated jumps of a function $f \in C^{-1}$.

For a fixed $r \in N$ and $f \in C^{-1}$ we set

$$IR_n(r;f;\cdot) \equiv \frac{r\pi}{d_r} n^r \begin{cases} (-1)^{((r+1)/2)} R_{n,2n}^{(-r)}(f;\cdot), & \text{if } r \text{ is odd}, \\ (-1)^{r/2+1} \tilde{R}_{n,2n}^{(-r)}(f;\cdot), & \text{if } r \text{ is even}, \end{cases} \tag{8}$$

where

$$d_r \equiv 1 - 2^{-r}.$$

For a fixed $r \in N$ and $M \in N$, the points $x_m(r; f; n)$, $m = 0, 1, \dots, M-1$, are defined via the following condition:

$$|IR_n(r; f; x_m(r; f; n))| = \max\{|IR_n(r; f; x)| : x \in B(x_m; \Delta_m(f))\}, \tag{9}$$

where $B(x_m; \Delta_m(f))$ is the closed ball around x_m with the radius $\Delta_m(f) = (1/3)\min\{|x_m - x_k| \bmod 2\pi : k = 0, 1, \dots, M-1 \text{ and } k \neq m\}$.

To simplify notations, we sometimes omit fixed parameters and write $x_m(n)$ or $x_m(r; n)$. Similarly we simplify the notation for $IR_n(r; f; \cdot)$.

LEMMA 1. *Let $s \in Z$. Then the following expansion holds for every fixed $a \in N$:*

$$\sum_{k=n}^{2n-1} \frac{1}{k^s} = \frac{\mathcal{A}_1(s)}{n^{s-1}} + \frac{\mathcal{A}_2(s)}{n^s} + \dots + \frac{\mathcal{A}_{a+1}(s)}{n^{s+a-1}} + O\left(\frac{1}{n^{s+a}}\right), \tag{10}$$

where the right-hand side has only a finite number of terms if $s \le 0$.

In particular

$$\mathcal{A}_1(s) = \frac{d_{s-1}}{s-1} \qquad \text{and} \qquad \mathcal{A}_2(s) = \frac{d_s}{2}, \tag{11}$$

for $s \ge 0$, where

$$\mathcal{A}_1(1) = \lim_{s \to 1} \mathcal{A}_1(s) = \ln 2. \tag{12}$$

PROOF. The proof for the case $s \le 0$ follows from the corresponding formula in [15, p. 1].

If $s \ge 2$, then (10) follows from the asymptotic expansion of the generalized zeta function $\zeta(s, n)$ (see [16, pp. 22 and 25]):

$$\sum_{k=n}^{\infty} \frac{1}{k^s} = \zeta(s, n) = \frac{1}{\Gamma(s)} \left(\frac{\Gamma(s-1)}{n^{s-1}} + \frac{\Gamma(s)}{2n^s} + \sum_{m=1}^{a-1} \frac{B_{2m}}{(2m)!} \frac{\Gamma(s+2m-1)}{n^{2m+s-1}} \right) + O\left(\frac{1}{n^{2a+s+1}}\right), \tag{13}$$

for $n \to \infty$ and a fixed $a \in N$. (Here Γ is the gamma function [16, p. 1] and B_m, $m \in N$, are Bernoulli numbers [16, p. 25].) To complete the proof for this case we just mention that $\Gamma(s) = (s-1)\Gamma(s-1)$, $s > 1$, [16, p. 2].

Meanwhile, it is known [16, p. 13] that if $\psi \equiv \Gamma'/\Gamma$ is the logarithmic derivative of the gamma function, then [16, p. 15]

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}, \tag{14}$$

for $n \in N$, where γ is Euler's constant, and [16, p. 18]

$$\psi(n) = \ln n - \frac{1}{2n} - \sum_{m=1}^{a} \frac{B_{2m}}{2m} \frac{1}{n^{2m}} + O\left(\frac{1}{n^{2a+2}}\right), \tag{15}$$

for $n \to \infty$ and a fixed $a \in N$.

Now asymptotic formula (10) for the case $s = 1$ is a simple combination of (14) and (15). ∎

The following are some basic properties of the functions $D_{n,2n}^{(-r)}$ and $\tilde{D}_{n,2n}^{(-r)}$, $r \in N$.

LEMMA 2. *Let $y_n \equiv y_n(r) > 0$ ($\tilde{y}_n \equiv \tilde{y}_n(r) > 0$) and $z_n \equiv z_n(r) > 0$ ($\tilde{z}_n \equiv \tilde{z}_n(r) > 0$) be the closest nonzero roots to the point zero of the equations $D_{n,2n}^{(-2r-2)}(x) = 0$ ($\tilde{D}_{n,2n}^{(-2r-1)}(x) = 0$) and $D_{n,2n}^{(-2r-1)}(x) = 0$ ($\tilde{D}_{n,2n}^{(-2r)}(x) = 0$), respectively. Then for any fixed $r \in Z_+$ we have the following.*

1. $y_n \in (\pi/4n, \pi/2n)$ ($\tilde{y}_n \in (\pi/4n, \pi/2n)$).
2. $z_n \in (\pi/2n, \pi/n)$ ($\tilde{z}_n \in (\pi/2n, \pi/n)$).
3. $(-1)^r D_{n,2n}^{(-2r-1)}(y_n) \simeq n^{-2r}$ ($(-1)^r \tilde{D}_{n,2n}^{(-2r)}(\tilde{y}_n) \simeq n^{-2r+1}$).
4. $(-1)^r D_{n,2n}^{(-2r-1)}$ ($(-1)^r \tilde{D}_{n,2n}^{(-2r)}$), *is increasing on* $[-y_n(r-1), y_n(r-1)]$ ($[-\tilde{y}_n(r-1), \tilde{y}_n(r-1)]$), *concave on* $[-y_n(r-1), 0]$ ($[-y_n(r-1), 0]$) *and convex on* $[0, y_n(r-1)]$ ($[0, \tilde{y}_n(r-1)]$).
5. $(-1)^{r+1} D_{n,2n}^{(-2r-2)}$ ($(-1)^{r+1} \tilde{D}_{n,2n}^{(-2r-1)}$) *is a 2π-periodic even and analytic function with the global maximum attained at $x = 2\pi k$, $k \in Z$. The sequence of the local maxima of $|D_{n,2n}^{(-2r-2)}|$ ($|\tilde{D}_{n,2n}^{(-2r-1)}|$) is decreasing as a function of $x \in [0, \pi]$. In addition, there exists a constant $\mathcal{K}(r) > 1$ ($\tilde{\mathcal{K}}(r) > 1$) such that*

$$\left| D_{n,2n}^{(-2r-2)}(0) \right| > \mathcal{K}(r) \left| D_{n,2n}^{(-2r-2)}(z_n) \right|, \tag{16}$$

$$\left(\left| \tilde{D}_{n,2n}^{(-2r-1)}(0) \right| > \tilde{\mathcal{K}}(r) \left| \tilde{D}_{n,2n}^{(-2r-1)}(\tilde{z}_n) \right| \right), \tag{17}$$

for $n > 1$.

PROOF.

1. By (6) we have

$$\operatorname{sign} D_{n,2n}^{(-2r-2)} \left(\frac{\pi}{2n} \right) = \operatorname{sign} (-1)^r. \tag{18}$$

However, by the same (6), $\operatorname{sign} D_{n,2n}^{(-2r-2)}(x) = \operatorname{sign} (-1)^{r+1}$ for any $x \in [0, \pi/4n]$. Now, the latter combined with (18) and the Mean Value Theorem obviously implies that $y_n \in (\pi/4n, \pi/2n)$.

2. The statement is proved analogously and we omit the details.
3. According to (6) and (10) we have

$$(-1)^r D_{n,2n}^{(-2r-1)}(x) = \sum_{k=n}^{2n-1} \frac{\sin kx}{k^{2r+1}} < \sum_{k=n}^{2n-1} \frac{1}{k^{2r+1}} \simeq n^{-2r}, \tag{19}$$

for $x \in R$. Meanwhile, since $y_n \in [\pi/4n, \pi/2n]$ (see Statement 1), taking into account the trivial inequality $2\sqrt{2}x/3\pi \leq \sin x$, valid for $x \in [0, 3\pi/4]$, (here $y = 2\sqrt{2}x/3\pi$ is the equation of the line passing through the origin and the point $((3\pi/4), \sin(3\pi/4))$) we have

$$\sum_{k=n}^{2n-1} \frac{\sin(ky_n)}{k^{2r+1}} > \frac{2\sqrt{2}}{3\pi} \sum_{k=n}^{[3n/2]} \frac{y_n}{k^{2r}} > \frac{\sqrt{2}}{6n} \sum_{k=n}^{[3n/2]} \frac{1}{k^{2r}} \simeq n^{-2r}, \tag{20}$$

where $[a]$ means the integer part of a number a. The combination of (19) and (20) completes the proof of Statement 3.

4. Since the function $(-1)^r D_{n,2n}^{(-2r)}$ is positive on $[-y_n(r-1), y_n(r-1)]$ (see (6)), $(-1)^r D_{n,2n}^{(-2r-1)}$ is increasing on the interval. Furthermore, $(-1)^r D_{n,2n}^{(-2r+1)}$ is positive and negative on $[-z_n(r-1), 0]$ and $[0, z_n(r-1)]$, respectively. Besides, $y_n(r-1) < z_n(r-1)$ (see Statements 1 and 2). Hence, $(-1)^r D_{n,2n}^{(-2r-1)}$ is concave and convex on $[-y_n(r-1), 0]$ and $[0, y_n(r-1)]$, respectively.
5. Let us prove inequality (16) as the rest of the statement is trivial. Let $\zeta_n \equiv (2n/\pi) z_n$. Then by Statement 2, $\zeta_n \in [1, 2]$ for $n \in N$. It is clear that

$$q_n(\zeta) \equiv \frac{\left| D_{n,2n}^{(-2r-2)}(0) \right|}{\left| D_{n,2n}^{(-2r-2)}(\zeta(\pi/2n)) \right|} = \sum_{k=n}^{2n-1} \frac{1}{k^{2r+2}} \left(\sum_{k=n}^{2n-1} \frac{\cos \zeta(k\pi/2n)}{k^{2r+2}} \right)^{-1} > 1, \tag{21}$$

for $\zeta \in [1,2]$ and $n > 1$. But

$$\lim_{n\to\infty} q_n(\zeta) = \lim_{n\to\infty} \frac{(\pi/2n) \sum\limits_{k=n}^{2n-1} (k\pi/2n)^{-2r-2}}{(\pi/2n) \sum\limits_{k=n}^{2n-1} \cos\left(\zeta(k\pi/2n)\right)(k\pi/2n)^{-2r-2}} = \frac{\int_{\pi/2}^{\pi} \left(1/t^{2r+2}\right) dt}{\int_{\pi/2}^{\pi} \left(\cos\zeta t/t^{2r+2}\right) dt} \equiv q(\zeta) \tag{22}$$

uniformly with respect to $\zeta \in [1,2]$ and obviously

$$\inf_{\zeta\in[1,2]} q(\zeta) > 1. \tag{23}$$

The rest instantly follows from (21)–(23).

The statements for $\tilde{D}_{n,2n}^{(-r)}$ are proved analogously, and we omit the details. ∎

3. MAIN IDENTITIES

The identity determining the jumps of a function of bounded variation by means of the partial sums of its differentiated Fourier series has been known for a long time.

THEOREM 1. *(See [17,18].) Let $f \in V$. Then the identity*

$$\lim_{n\to\infty} \frac{S_n'(f;x)}{n} = \frac{1}{\pi}\left(f(x+) - f(x-)\right) \tag{24}$$

is valid for each fixed $x \in [-\pi,\pi]$.

Golubov generalized identity (24) for the V_p classes of functions and higher derivatives of the partial sums of Fourier and the series conjugate to the Fourier series.

THEOREM 2. *(See [19].) Let $r \in Z_+$ and suppose $f \in V_p$ for some $p \geq 1$. Then*

1. *the identity*
$$\lim_{n\to\infty} \frac{S_n^{(2r+1)}(f;x)}{n^{2r+1}} = \frac{(-1)^r}{(2r+1)\pi}\left(f(x+) - f(x-)\right)$$
is valid for each fixed $x \in [-\pi,\pi]$;
2. *there is no way to determine the jump at the point $x \in [-\pi,\pi]$ of an arbitrary function $f \in V_p$, $p \geq 1$, by means of the sequence $(S_n^{(2r)}(f;\cdot))_{n=0}^{\infty}$.*

THEOREM 3. *(See [19].) Let $r \in N$ and suppose $f \in V_p$ for some $p \geq 1$. Then*

1. *the identity*
$$\lim_{n\to\infty} \frac{\tilde{S}_n^{(2r)}(f;x)}{n^{2r}} = \frac{(-1)^{r+1}}{2r\pi}\left(f(x+) - f(x-)\right)$$
is valid for each fixed $x \in [-\pi,\pi]$;
2. *there is no way to determine the jump at the point $x \in [-\pi,\pi]$ of an arbitrary function $f \in V_p$, $p \geq 1$, by means of the sequence $(\tilde{S}_n^{(2r-1)}(f;\cdot))_{n=1}^{\infty}$.*

Further generalizations, extending the results of Golubov to V_Φ, ΛBV, and $V[v]$ classes of functions have been obtained by one of the authors [20]. (For the definitions of these classes of functions see [21–23], respectively.)

Now we show that similar identities hold if we consider the integrated rather than the differentiated Fourier series.

THEOREM 4. *Let $r \in Z_+$ and suppose the function $f \in V_p$, $1 \le p < 2$, has a finite number of discontinuities. Then*

1. *the identity*

$$\lim_{n\to\infty} n^{2r+1} R_n^{(-2r-1)}(f;x) = \frac{(-1)^{r+1}}{(2r+1)\pi}\left(f(x+)-f(x-)\right) \tag{25}$$

is valid for each fixed $x \in [-\pi,\pi]$;

2. *there is no way to determine the jump at the point $x \in [-\pi,\pi]$ of an arbitrary function $f \in V_p$, $p \ge 1$, by means of the sequence $(R_n^{(-2r-2)}(f;\cdot))_{n=1}^{\infty}$.*

THEOREM 5. *Let $r \in N$ and suppose the function $f \in V_p$, $1 \le p < 2$, has a finite number of discontinuities. Then*

1. *the identity*

$$\lim_{n\to\infty} n^{2r} \tilde{R}_n^{(-2r)}(f;x) = \frac{(-1)^{r+1}}{2r\pi}\left(f(x+)-f(x-)\right) \tag{26}$$

is valid for each fixed $x \in [-\pi,\pi]$;

2. *there is no way to determine the jump at the point $x \in [-\pi,\pi]$ of an arbitrary function $f \in V_p$, $p \ge 1$, by means of the sequence $(\tilde{R}_n^{(-2r-1)}(f;\cdot))_{n=1}^{\infty}$.*

PROOF OF THEOREM 4.

1. It is known [14, p. 75] that

$$V_p \subset C^{-1} \tag{27}$$

for any $p \ge 1$. So, the Fourier series of the function $f \in V_p$ is well defined.

Obviously, by means of a change of variables, the problem can always be reduced to the case $x = 0$.

Now, according to (6) and (13) we have:

$$\lim_{n\to\infty} n^{2r+1} R_n^{(-2r-1)}(G;0) = \frac{(-1)^{r+1}}{(2r+1)}. \tag{28}$$

Next, for the given function $f \in V_p$, let us set

$$f_c \equiv f - \frac{1}{\pi}\sum_{m=0}^{M-1} [f]_m G(x_m;\cdot), \tag{29}$$

where $x_0 = 0, x_1, \dots, x_{M-1}$, and $[f]_m$, $m = 0,1\dots,M-1$, are the points of discontinuity and the associated jumps of the function f.

Obviously,

$$f_c \in C \cap V_p. \tag{30}$$

Continuity of f_c follows from (29). Besides, since $G \in V \subset V_p$ and V_p is a linear vector space, $f_c \in V_p$ as well.

It is known that if $f \in V_p$, $1 \le p < 2$, then the function f is continuous if and only if its Fourier coefficients satisfy the following condition [24]:

$$\sum_{k=n}^{\infty} \left(a_k(f)^2 + b_k(f)^2\right) = o\left(\frac{1}{n}\right). \tag{31}$$

Thus, according to (13), (30), (31), and Cauchy-Schwartz inequality we have:

$$\begin{aligned} n^{2r+1}\left|R_n^{(-2r-1)}(f_c;x)\right| &\le n^{2r+1}\sum_{k=n}^{\infty}\frac{|a_k(f_c)|+|b_k(f_c)|}{k^{2r+1}} \\ &\le \sqrt{2}\,n^{2r+1}\left(\sum_{k=n}^{\infty}\left(a_k(f_c)^2+b_k(f_c)^2\right)\right)^{1/2}\left(\sum_{k=n}^{\infty}\frac{1}{k^{4r+2}}\right)^{1/2} \\ &= n^{2r+1}o\left(n^{-(1/2)}\right)O\left(n^{-2r-(1/2)}\right)=o(1) \end{aligned} \tag{32}$$

uniformly with respect to $x \in [-\pi, \pi]$.

Besides, by virtue of (5), (13), and Abel's transformation we have:

$$\begin{aligned} \left|\sum_{k=n}^{\infty}\frac{\cos k\beta}{k^s}\right| &\le \left|\sum_{k=n}^{n^2-1}\left(\frac{1}{k^s}-\frac{1}{(k+1)^s}\right)\sum_{j=n}^{k}\cos j\beta+\frac{1}{n^{2s}}\sum_{k=n}^{n^2}\cos k\beta\right|+\sum_{k=n^2+1}^{\infty}\frac{1}{k^s} \\ &\le \frac{\mathcal{K}}{\beta}\frac{1}{n^s}+O\left(\frac{1}{n^{2s-1}}\right)=\frac{1}{\beta}O\left(\frac{1}{n^s}\right), \end{aligned} \tag{33}$$

for any $0 < \beta < \pi$ and $s \ge 2$, which taking into account (6) proves that

$$n^{2r+1}R_n^{(-2r-1)}(G(x_m;\cdot);0)=\frac{1}{\Delta_m}O\left(\frac{1}{n}\right)=o(1), \tag{34}$$

for $m = 1, 2, \dots, M-1$. Now, a combination of (28), (29), (32), and (34) completes the proof.

2. As to assertion 2 of Theorem 4, for any odd integrable function f, $R_n^{(-2r-2)}(f;0) = 0$, $r, n \in N$, independent of the existence of a jump of the function f at $x = 0$. ∎

Theorem 5 is proved virtually identically, and so we omit the details.

4. GENERAL IDEA OF THE ALGORITHMS

The following is the general idea of the algorithm: according to identities (25) and (26), if a function $f \in V_p$, $1 \le p < 2$, has a finite number of discontinuities, then for a fixed $r \in Z_+$ and sufficiently large $n \in N$, the function $|R_n^{(-2r-1)}(f;\cdot)|$ (or $|\tilde{R}_n^{(-2r-2)}(f;\cdot)|$), $x \in [-\pi,\pi]$, must attain the largest local maximum nearby the actual points of discontinuity of the function f, since at the points of continuity of f, $R_n^{(-2r-1)}(f;x) = o(1)$ by virtue of Theorem 4. Hence, we could search for the singularity locations of the function by finding the largest local spikes of the integrated tails of its Fourier series. However, according to our main assumption, only a *finite* number of the Fourier coefficients are known. Fortunately, this does not represent a major obstacle since we can consider *truncated* tails of the Fourier series or the conjugate to the Fourier series of the function. This change will result in only a scaling of formulae (25) and (26). Namely, under the conditions analogous to those of Theorem 4 and Theorem 5, we have:

$$\lim_{n\to\infty} n^{2r+1}R_{n,2n}^{(-2r-1)}(f;x)=\frac{(-1)^{r+1}d_{2r+1}}{(2r+1)\pi}\left(f(x+)-f(x-)\right) \tag{35}$$

and

$$\lim_{n\to\infty} n^{2r}\tilde{R}_{n,2n}^{(-2r)}(f;x)=\frac{(-1)^{r+1}d_{2r}}{2r\pi}\left(f(x+)-f(x-)\right). \tag{36}$$

Figures 1–4 represent the graphs of the normalized integrated truncated tails $-2\pi n R_{n,2n}^{(-1)}(f;\cdot)$ of the function (51). They illustrate the dynamics of creation of sharp spikes in the vicinity of the actual points of discontinuities of the function.

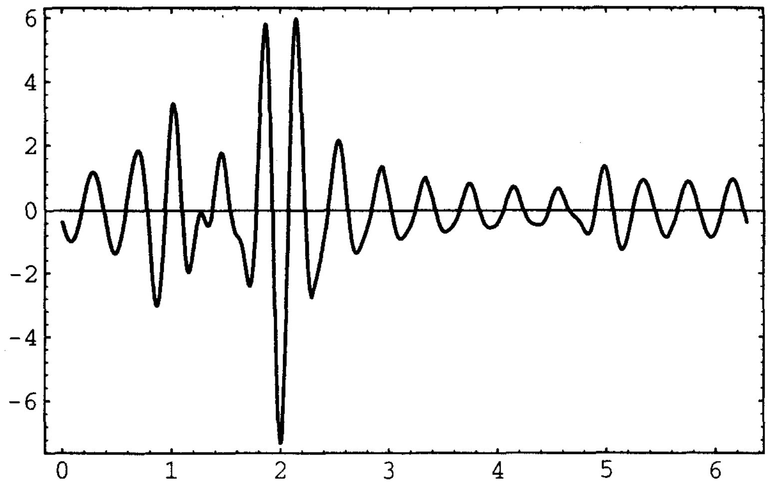

Figure 1. Graph of the normalized integrated Fourier truncated tail for $n = 16$ of function (51).

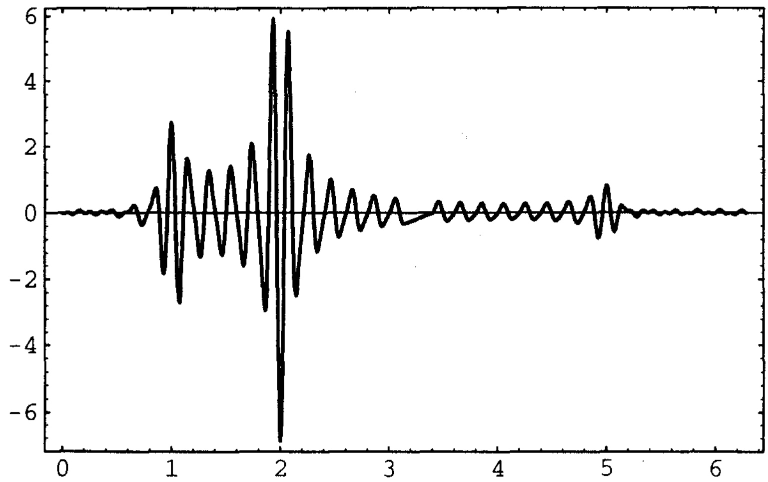

Figure 2. Graph of the normalized integrated Fourier truncated tail for $n = 32$ of function (51).

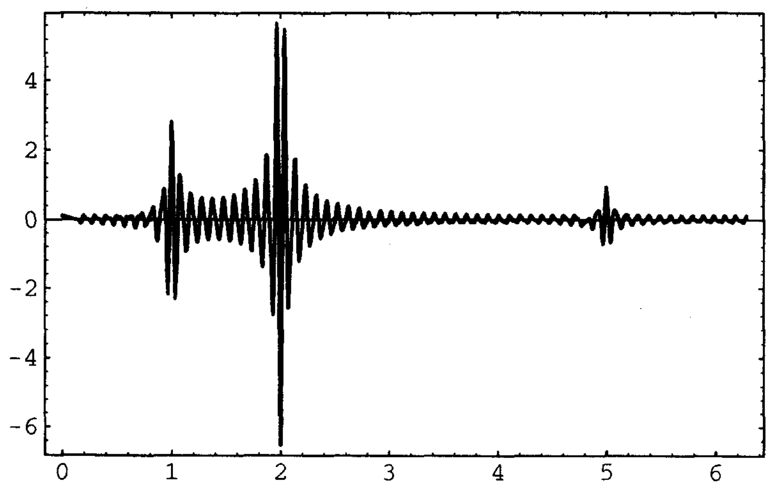

Figure 3. Graph of the normalized integrated Fourier truncated tail for $n = 64$ of function (51).

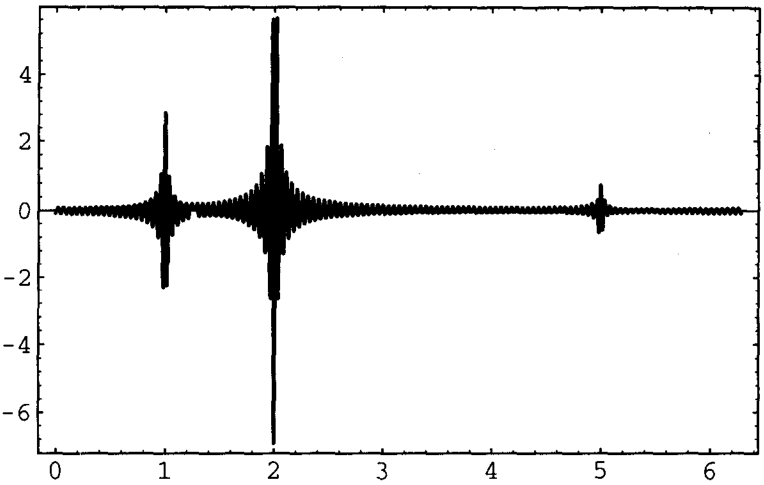

Figure 4. Graph of the normalized integrated Fourier truncated tail for $n = 128$ of function (51).

5. APPROXIMATION TO THE POINTS OF DISCONTINUITY AND THE ASSOCIATED JUMPS

Now we consider how well the points $x_m(n)$ and the values $IR_n(x_m(n))$ approximate the points of discontinuity x_m and the jumps $[f]_m$ of the function f.

Let us start from the most general case.

THEOREM 6. *Let $r \in N$ be fixed, and suppose the function $f \in V_p$, $1 \leq p < 2$, has a finite number, $M(f)$, of discontinuities. Then the estimate*

$$x_m(r; f; n) = x_m(f) + \frac{1}{[f]_m \Delta_m(f)} o\left(\frac{1}{n}\right) \tag{37}$$

is valid for each fixed $m = 0, 1, \dots, M(f) - 1$.

Let us now consider functions with more smoothness.

THEOREM 7. *Let $r \in N$ be fixed, and suppose f is the piecewise continuous function such that $f' \in V_p$, $1 \leq p < 2$. In addition, we assume that $M(f)$ and $M(f')$ are finite. Then the estimate*

$$x_m(r; f; n) = x_m(f) + \frac{1}{[f]_m} \left(\left[f'\right]_m O\left(\frac{1}{n^2}\right) + \sum_{k \neq m} \frac{[f]_k}{\Delta_k(f)} O\left(\frac{1}{n^2}\right) \right) \tag{38}$$

is valid for each $m = 0, 1, \dots, M(f) - 1$.

Finally, we consider probably the most interesting case: a 2π-periodic piecewise smooth function with one jump discontinuity. As expected, the approximation in this case is significantly more regular. Namely, the following statement holds.

THEOREM 8. *Let $r \in N$ be fixed, and suppose the function f piecewise belongs to C^q, $q \geq 2$, and has a single discontinuity at $x_0(f) \in (-\pi, \pi)$. In addition, we assume that $f^{(q)} \in V_p$, $1 \leq p < 2$. Then there exist constants $\mathcal{K}_i \equiv \mathcal{K}_i(r) \equiv \mathcal{K}_i(r; f)$, $i = 1, 2, \dots q$, such that*

$$x_0(r; f; n) = x_0(f) + \frac{\mathcal{K}_1(r)}{n^2} + \frac{\mathcal{K}_2(r)}{n^3} + \cdots + \frac{\mathcal{K}_q(r)}{n^{q+1}} + o\left(\frac{1}{n^{q+1}}\right). \tag{39}$$

Namely,

$$\mathcal{K}_1(r) = \frac{d_r}{d_{r-2}} \frac{r-2}{r} \frac{\left[f'\right]_0}{[f]_0}, \tag{40}$$

and

$$\mathcal{K}_2(r) = \frac{1}{2}\left(r\,\frac{d_{r+1}}{d_r} - (r-2)\,\frac{d_{r-1}}{d_{r-2}}\right)\frac{d_r}{d_{r-2}}\,\frac{r-2}{r}\,\frac{\left[f'\right]_0}{[f]_0}, \tag{41}$$

for $r \in N$, under assumption (12) for $r = 2$.

Taking advantage of the explicit knowledge of the first two coefficients (40) and (41), using a simple linear combination of expansion (39) for distinct integrals $1 \le r_1 < r_2 < r_3$, we are able to significantly improve the accuracy of the initial approximation. Namely, the following statement holds.

COROLLARY 1. *Suppose the function f piecewise belongs to C^q, $q \ge 3$, and has a single discontinuity at $x_0(f) \in (-\pi, \pi)$. In addition, we assume that $f^{(q)} \in V_p$, $1 \le p < 2$. Then for each fixed triple $1 \le r_1 < r_2 < r_3$ there exist constants $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$, independent of n, such that*

$$\sum_{i=1}^{3} \mathcal{B}_i x_0(r_i; f; n) = x_0(f) + \frac{\mathcal{K}_1}{n^4} + \cdots + \frac{\mathcal{K}_{q-2}}{n^{q+1}} + o\left(\frac{1}{n^{q+1}}\right), \tag{42}$$

where $\mathcal{K}_i \equiv \mathcal{K}_i(r_1; r_2; r_3; f)$, $i = 1, 2, \ldots, q-2$, are fixed constants.

Below, we present a table of constants $\mathcal{B}_i$, $i = 1, 2, 3$, for some choices of r_i.

Table 1. Table for the constants $\mathcal{B}_i$ for some choices of r_i.

r_1	r_2	r_3	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$
1	3	5	−231	477	−245
3	4	5	−216	462	−245

Now let us study the approximation to the jumps of a function.

THEOREM 9. *Let $r \in N$ be fixed, and suppose the function f is a piecewise continuous function such that $f' \in V_p$, $1 \le p < 2$. In addition, we assume that $M(f)$ and $M(f')$ are finite. Then the estimate*

$$IR_n(r; f; x_m(n)) = [f]_m + O\left(\frac{1}{n}\right) \tag{43}$$

is valid for each $m = 0, 1, \ldots, M(f) - 1$.

THEOREM 10. *Let $r \in N$, and suppose the function f piecewise belongs to C^q, $q \ge 2$, and has a single discontinuity at $x_0(f) \in (-\pi, \pi)$. In addition, we assume that $f^{(q)} \in V_p$, $1 \le p < 2$. Then there exist constants $\mathcal{K}_i \equiv \mathcal{K}_i(r; f)$, $i = 1, 2, \ldots, q$, such that*

$$IR_n(r; f; x_0(n)) = [f]_0 + \frac{\mathcal{K}_1}{n} + \frac{\mathcal{K}_2}{n^2} + \cdots + \frac{\mathcal{K}_q}{n^q} + o\left(\frac{1}{n^q}\right). \tag{44}$$

Namely,

$$\mathcal{K}_1 = \frac{d_{r+1}}{d_r}\,\frac{r}{2}\,[f]_0, \tag{45}$$

for $r \in N$.

Extrapolating expansion (44) based on identity (45), we improve the accuracy of the initial approximation. Namely, the following statement holds.

COROLLARY 2. *Let $r \in N$ be fixed, and suppose the function f piecewise belongs to C^q, $q \ge 3$, and has a single discontinuity at $x_0(f) \in (-\pi, \pi)$. In addition, we assume that $f^{(q)} \in V_p$, $1 \le p < 2$. Then for a fixed $1 \le r_1 < r_2$, there exist constants $\mathcal{C}_1$ and $\mathcal{C}_2$ such that*

$$\sum_{i=1}^{2} \mathcal{C}_i IR_n(r_i; f; x_0(n)) = [f]_0 + \frac{\mathcal{K}_1}{n^2} + \cdots + \frac{\mathcal{K}_{q-1}}{n^q} + o\left(\frac{1}{n^q}\right), \tag{46}$$

where $\mathcal{K}_i \equiv \mathcal{K}_i(r_1; r_2; f)$, $i = 1, 2, \ldots, q-1$, are fixed constants.

Namely,

$$\mathcal{C}_1 = \frac{r_2 d_{r_2+1}}{2d_{r_2}} \left(\frac{r_2 d_{r_2+1}}{2d_{r_2}} - \frac{r_1 d_{r_1+1}}{2d_{r_1}} \right)^{-1} \tag{47}$$

and

$$\mathcal{C}_2 = -\frac{r_1 d_{r_1+1}}{2d_{r_1}} \left(\frac{r_2 d_{r_2+1}}{2d_{r_2}} - \frac{r_1 d_{r_1+1}}{2d_{r_1}} \right)^{-1}. \tag{48}$$

For the proofs of Theorems 6–10 we refer the reader to [25], in particular to the proofs of Theorems 4–8, respectively. Taking into consideration Lemma 1 and 2 they are completely analogous.

6. DESCRIPTION OF THE ALGORITHM

We now describe the algorithm we have implemented to locate the points of discontinuity. For simplicity, we assume that the function is piecewise infinitely differentiable.

In case the function has a single discontinuity, according to Corollary 1 we search for the global maximum of $|IR_n|$ for fixed $1 \le r_1 < r_2 < r_3$. Afterwards, utilizing expansion (42) and (46), and applying Richardson's method of extrapolation, we improve the accuracy.

Unfortunately, expansion formula (39) does hot hold for functions with multiple discontinuities. We overcome this difficulty using the techniques we introduced in [13]. Precisely, we generate, for a fixed $r \in N$, the sequence of truncated tails of Fourier series of functions $(f_m)_{m=0}^{M-1}$, defined via the recursion relation

$$f_m(x) = f(x) \prod_{k=0,k\neq m}^{M-1} (1 - \cos(x - x_k(f;n)))^d, \tag{49}$$

where $d \in N$ is fixed, and $x_k(f;n)$, $k = 0, 1, \dots, M-1$, is an initial approximation to $x_k(f)$. The idea is that by multiplying a function f by the factor $\prod_{k=0,k\neq m}^{M-1}(1 - \cos(x - x_k(f;n)))^d$ we are not adding a new point of discontinuity but significantly reducing the jumps at every discontinuity point except at x_m. More precisely, if $x_k - x_k(f;n) = O(n^{-\alpha})$, $k = 0, 1, \dots, M-1$, then the function f_m (see (49)) at the point θ_k, $k \neq m$, will have a jump of order $(1 - \cos(x_k - x_k(f;n)))^d [f_k]_0 \simeq n^{-2d\alpha}[f_k]_0$. More details are given in [13]. We note that this method a treating the multiple discontinuity case could also be applied in conjunction with other algorithms for discontinuity location.

Summarizing all the above we suggest an algorithm which, with obvious changes, is essentially analogous to the one described in [13].

7. NUMERICAL RESULTS

In order to illustrate the numerical results obtained by the described algorithm, we will consider several examples.

The following piecewise smooth function was considered in [1].

$$f(x) = \begin{cases} \sin\dfrac{x}{2}, & \text{if } 0 \le x \le 0.9, \\ -\sin\dfrac{x}{2}, & \text{if } 0.9 < x < 2\pi. \end{cases} \tag{50}$$

By applying the suggested method the authors obtained the errors 2.6(−4) and 2.7(−5) in the estimates of the discontinuity location and the associated jump for $n = 128$, respectively.

Below we present a detailed description of all computations. The first table shows the error in the approximation to the location of the discontinuity by the integrated truncated tails of its Fourier series of degree $r_1 = 1$, $r_2 = 3$, and $r_3 = 5$, and then their linear combination via formula (42).

Table 3 presents a complete calculation of Richardson's extrapolation started from the last column of Table 2.

Table 4 presents the error in the approximation to the jumps of the function using $r_1 = 1$ and $r_2 = 3$, and their combination (46).

Table 5 is Richardson's extrapolation applied to data of Table 4.

Table 2. Initial approximations to the location of discontinuity and their linear combination by (42).

n	$r_1 = 1$	$r_2 = 3$	$r_3 = 5$	Linear Combination by (42)
4	3.85(−2)	4.54(−2)	5.14(−2)	1.31(−1)
8	8.86(−3)	1.04(−2)	1.19(−2)	7.62(−3)
16	2.11(−3)	2.48(−3)	2.83(−3)	4.33(−4)
32	5.17(−4)	6.05(−4)	6.90(−4)	2.54(−5)
64	1.27(−4)	1.49(−4)	1.70(−4)	1.54(−6)
128	3.17(−5)	3.70(−5)	4.22(−5)	9.45(−8)

Table 3. Complete table of Richardson's extrapolation for the location of discontinuity of function (50) using the integrated Fourier series.

n	$r_1, r_2, r_3 = 1, 3, 5$					
4	1.3(−1)					
8	7.6(−3)	6.4(−4)				
16	4.3(−4)	4.6(−5)	2.7(−5)			
32	2.5(−5)	1.8(−6)	2.9(−7)	1.3(−7)		
64	1.5(−6)	5.7(−8)	3.1(−9)	1.4(−9)	3.5(−10)	
128	9.4(−8)	1.8(−9)	3.7(−11)	1.1(−11)	4.1(−13)	9.7(−13)

Table 4. Initial approximations to the magnitude of jump and their linear combination by (46).

n	$r_1 = 1$	$r_2 = 3$	Linear Combination by (46)
4	2.11(−1)	4.54(−1)	8.40(−4)
8	9.23(−2)	1.98(−1)	5.88(−4)
16	4.33(−2)	9.30(−2)	1.86(−4)
32	2.10(−2)	4.50(−2)	5.10(−5)
64	1.03(−2)	2.21(−2)	1.32(−5)
128	5.13(−3)	1.10(−2)	3.38(−6)

Table 5. Complete table of Richardson's extrapolation for the magnitude of the jump of function (50) using the integrated Fourier series.

n	$r_1, r_2 = 1, 3$					
4	8.4(−4)					
8	5.8(−4)	5.0(−4)				
16	1.8(−4)	5.2(−5)	1.2(−5)			
32	5.1(−5)	5.9(−6)	7.1(−7)	5.3(−8)		
64	1.3(−5)	7.0(−7)	4.4(−8)	8.3(−10)	8.3(−10)	
128	3.3(−6)	8.6(−8)	2.7(−9)	2.3(−11)	2.5(−12)	1.0(−11)

The next example has been considered in [7]:

$$f(x) = \begin{cases} 0, & \text{if } 0 < x \leq 1, \\ e^x, & \text{if } 1 < x \leq 2, \\ \cos\dfrac{x}{2}, & \text{if } 2 < x \leq 5, \\ 0, & \text{if } 5 < x \leq 2\pi. \end{cases} \tag{51}$$

This function has been used by all other researchers in order to illustrate their method. Below we present the absolute value of the largest error in the estimation of the points of discontinuity of function (51), obtained by Bauer, Eckhoff, Banerjee and Geer, and by us, summarized in Tables 6–10.

Table 6. Largest errors in the estimates of the discontinuity locations for function (51) using Bauer's method.

n	32	64	128	256
Local Filter	1.4(−5)	1.4(−6)	5.6(−9)	3.1(−9)
Global Filter	1.5(−3)	8.7(−5)	2.2(−6)	2.7(−7)

Table 7. Largest errors in the estimates of the discontinuity locations for function (51) using various Eckhoff's methods.

n	32	64	128	256
$q = 0$	1.4(−3)	3.1(−4)	7.3(−5)	1.8(−5)
$q = 1$	2.2(−5)	1.0(−6)	5.6(−8)	2.7(−9)
$q = 2$	4.0(−6)	1.2(−7)	5.1(−9)	2.7(−10)
$q = 3$	1.2(−7)	2.8(−10)	5.3(−11)	4.4(−5)

Table 8. Largest errors in the estimates of the discontinuity locations for function (51) using various LSPE methods.

n	32	64	128	256
$q = 0$	1.5(−3)	3.3(−4)	7.7(−5)	1.7(−5)
$q = 1$	1.4(−4)	1.1(−6)	2.1(−7)	1.4(−8)
$q = 2$	2.9(−6)	1.3(−7)	7.1(−9)	4.2(−10)
$q = 3$	6.8(−8)	4.1(−9)	4.9(−11)	1.4(−13)

Table 9. Largest errors in the estimates to the discontinuity locations for function (51) using our method with differentiated Fourier series.

n	32	64	128	256	512
Location-Error	8.6(−5)	5.4(−7)	1.3(−8)	2.3(−10)	3.3(−14)

Table 10. Largest errors in the estimates to the discontinuity locations for function (51) using our method with integrated Fourier series.

n	32	64	128	256	512
Location-Error	4.0(−5)	3.9(−7)	3.3(−9)	6.5(−11)	3.3(−13)

8. STABILITY

In the present section, we study stability of the algorithm by introducing some random errors. Following Geer and Banerjee [9, p. 17] we consider again the function (51), but with Fourier coefficients contaminated by some random errors. We then apply our method using these contaminated coefficients, and compare the results with those obtained using the original coefficients. Here we define the new coefficients ($k = 1, 2, \dots, n$)

$$\bar{a}_k = a_k(f) + \epsilon_{a,k}, \qquad \bar{b}_k = b_k(f) + \epsilon_{b,k},$$

where $\epsilon_{a,k}$ and $\epsilon_{b,k}$ are independent, uniformly distributed random variables on the interval $[-\epsilon, \epsilon]$, with $\epsilon > 0$ specified.

The absolute errors in the estimates of the discontinuity locations, as well as the relative errors of the corresponding jumps of function (51), using our method with $n = 128$, are summarized in Tables 11–13. Table 14 represents similar computations utilizing the LSPE method. We considered 25 different contaminated coefficients for each choice of ϵ.

Table 11. Means of the errors in the estimates for the discontinuity location $x_1 = 1$ and the associated jump for function (51).

ϵ	Mean of Loc.-Er.	Std. Dev.	Mean of Jump-Er.	Std. Dev.
10^{-2}	1.8(−4)	3.2(−2)	1.5(−2)	2.0(−1)
10^{-3}	4.8(−5)	2.4(−3)	3.6(−3)	6.3(−2)
10^{-4}	6.4(−6)	2.4(−4)	4.4(−4)	2.3(−3)
0	3.3(−9)		2.0(−4)	

Table 12. Means of the errors in the estimates for the discontinuity location $x_2 = 2$ and the associated jump for function (51).

ϵ	Mean of Loc.-Er.	Std. Dev.	Mean of Jump-Er.	Std. Dev.
10^{-2}	5.1(−3)	2.3(−2)	1.8(−2)	1.6(−1)
10^{-3}	8.0(−4)	2.0(−3)	2.6(−3)	1.3(−2)
10^{-4}	8.1(−5)	2.0(−4)	2.5(−4)	1.3(−3)
0	2.9(−9)		3.0(−6)	

Table 13. Means of the errors in the estimates for the discontinuity location $x_3 = 5$ and the associated jump for function (51).

ϵ	Mean of Loc.-Er.	Std. Dev.	Mean of Jump-Er.	Std. Dev.
10^{-2}	1.2(−2)	1.0(−1)	2.6(−1)	6.4(−1)
10^{-3}	7.0(−4)	8.9(−3)	5.3(−3)	6.4(−2)
10^{-4}	8.0(−5)	8.9(−4)	4.6(−4)	6.4(−3)
0	4.9(−11)		1.8(−8)	

Table 14. Errors in the estimates for the discontinuity location $x_1 = 1$ and the associated jump for function (51) using the LSPE method with $q = 1$.

ϵ	Location-Error	Jump-Error
10^{-2}	7.5(−3)	1.0(−2)
10^{-3}	6.8(−4)	1.5(−3)
10^{-4}	6.9(−5)	2.0(−4)
0	1.9(−6)	3.9(−4)

We encountered a failure of the method only for $\epsilon = 10^{-2}$: eight times it failed to refine the discontinuity location for $\theta_3 = 5$ applying higher integrals of the Fourier partial sums. Let us mention that at $\theta_3 = 5$ the function has the smallest jump equal to 0.80114362.

9. COMPLEXITY AND TIMING RESULTS

Because of its flexibility and the presence of powerful built-in tools, our initial development of this algorithm was done in *Mathematica*. In particular we note the following.

(1) The adaptive two-dimensional plotting routine and graphics interface allows us to locate the spikes of the derivative quickly and to visually check that the algorithm is performing as expected.

(2) The arbitrary precision arithmetic package, which tracks loss of precision, guarantees that the answers are correct and uncorrupted by round-off error, that is, we know that we are observing errors introduced by the algorithm due to truncation of the infinite series and not errors introduced by other causes. (Our final results were, however, computed using standard floating point arithmetic.)

(3) The symbolic manipulation capability allows us to test functions expressed analytically (the Fourier coefficients are easily generated), functions whose location and size of discontinuities we know.

Again, we refer to [13] for a detailed discussion of the cost of various phases of the algorithm. Since, for Fourier series, both differentiation and integration correspond to essentially diagonal matrix operations, there are no fundamental differences. One difference is that three values of r are used here, while two are used in the differentiation implementation. As the initial phase uses a fixed number of points n_0, the analysis indicates a linear growth in n. Our timing results, in contrast, show nonlinear growth, which we attribute to memory access features of *Mathematica*. These were obtained on a 143 MHz SUN Ultra 1 Model 140 system with 192 M memory running Solaris 4.5.

Table 15. Running time of the refinement phase of the algorithm for function (51).

n	Running Time in Seconds
32	9.2
64	18.1
128	38.5
256	90.8
512	226.2

10. CONCLUSION

Let us give some comments on our results.

Although the formula which determines the jumps of a bounded not-too-highly oscillating function by means of derivatives of its Fourier series has been known for a long time, identities (25) and (26) are new, and in some sense, symmetric to those of Theorems 2 and 3. To our best knowledge they have never been utilized for a numerical approximation of the locations of discontinuity points. Theorems 4 and 5 imply that it is possible to detect the locations of discontinuities and the jumps under mild conditions on the function. (Of course, we do assume that the Fourier coefficients themselves are known.)

We conjecture that Theorem 4 and 5 hold for a function of V_p class for *any* $p \geq 1$, and for even larger classes of a generalized bounded variation, namely HBV classes [22]. But, since our main goal is to develop a method which is applicable to a piecewise smooth function, we do not bother the reader with technical details and plan to study the question elsewhere.

It follows from Theorem 6 that identities (35) and (36) can be used even when multiple discontinuities are present, with an error of order $o(1/n)$. The factor $([f]_m \Delta_m(f))^{-1}$ is not surprising:

the smaller the jump of a function and the distance between the points of singularity, the more difficult it is to detect its location.

Taking into consideration asymptotic expansions (39) and (44), we see that the most rapid convergence is obtained for a piecewise smooth function with a single discontinuity. Particularly useful and interesting are the expansion formulae (42) and (46), obtained via the combination of different integrals of the truncated Fourier sums. For piecewise smooth functions, we can obtain very high orders of approximation. The numerical results confirm that high accuracy is indeed attainable with fairly low degree trigonometric polynomials. Then, applying the suggested method of "removing" discontinuities, we obtain good results for a function with multiple discontinuities, too. It should be mentioned as well that increasing the order of the partial sums we obtained better results: for the same function (50) using expansion (42) and (46), $r_1 = 1$, $r_2 = 3$, and $r_3 = 5$, combined with Richardson's extrapolation for $n = 512$ we achieved an accuracy of 10^{-17} for the location of the point of discontinuity and the associated jump.

Numerical results obtained via differentiated and integrated Fourier series are within same range (see Tables 9 and 10). For the Fourier case, there is no particular reason to prefer one over the other. However, considering potential generalizations to Jacobi series, we believe that integration techniques may be preferable, due to the better properties of integration operators in these families.

Although we have developed a good method for coping with the breakdown of the expansion formulae in the case of multiple discontinuities, the other researchers suggested algorithms specifically designed for such functions. The methods suggested by Banerjee, Geer, and Eckhoff, do give slightly better numerical results than ours. However, our treatment of multiple discontinuities is automatic (though our techniques here might also be usable in conjunction with the other author's methods) and we have rigorous theoretical results. We also believe our approach has good potential for extensions to other bases and to multiple dimensions.

REFERENCES

1. W. Cai, D. Gottlieb and C.-W. Shu, Essentially nonoscillatory spectral Fourier methods for shock wave calculations, *Math. Comp.* **52**, 389–410, (1989).
2. S. Abarbanel and D. Gottlieb, Information content in spectral calculations, In *Progress in Scientific Computing, Vol. 6*, (Proc. U.S.-Israel Workshop, 1984), (Edited by E.M. Murman and S.S. Abarbanel), pp. 345–356, Birkhäuser, Boston, (1985).
3. S. Abarbanel, D. Gottlieb and E. Tadmor, Spectral methods for discontinuous problems, In *Numerical Methods in Fluid Dynamics*, II, Proc. Conf., Reading, 1985, (Edited by K.W. Morton and M.J. Baines), pp. 129–153, Clarendon Press, Oxford, (1986).
4. D. Gottlieb, Spectral methods for compressible flow problems, In *Proc. 9$^{\text{th}}$ Internat. Conf. Numer. Methods Fluid Dynamics*, Saclay, France, 1984, (Edited by Soubbaramayer and J.P. Boujot), Lecture Notes in Phys., Vol. 218, pp. 48-61, Springer-Verlag, Berlin, (1985).
5. D. Gottlieb, L. Lustman and S.A. Orszag, Spectral calculations of one-dimensional inviscid compressible flows, *SIAM J. Sci. Comput.* **2**, 296–310, (1981).
6. K.S. Eckhoff, Accurate and efficient reconstruction of discontinuous functions from truncated series expansions, *Math. Comp.* **61**, 745–763, (1993).
7. K.S. Eckhoff, Accurate reconstruction of functions of finite regularity from truncated Fourier series expansions, *Math. Comp.* **64**, 671–690, (1995).
8. R.B. Bauer, Numerical shock capturing techniques, Doctoral Thesis, Division of Applied Mathematics, Brown University, (1995).
9. N.S. Banerjee and J.F. Geer, Exponential approximations using Fourier series partial sums, ICASE Rep. No. 97-56.
10. J. Geer, Rational trigonometric approximations using Fourier series partial sums, *J. Sci. Comput.* **10**, 325–356, (1995).
11. J. Geer and N.S. Banerjee, Exponentally accurate approximations to piece-wise smooth periodic functions, ICASE Rep. No. 95-17.
12. A. Gelb and E. Tadmor, Detection of Edges in spectral data, *Appl. Comput. Harmon. Anal.* **7**, 101–135, (1999).
13. G. Kvernadze, T. Hagstrom and H. Shapiro, Locating the discontinuities of a bounded function by the partail sums of its Fourier series, *J. Sci. Comput.* (to appear).

14. N. Wiener, The quadratic variation of a function and its Fourier coefficients, *Journal of Mathematics and Physics* **3**, 72–94, (1924).
15. I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, New York, (1980).
16. W. Magnus, F. Oberhettinger and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Third Edition, Springer-Verlag, New York, (1966).
17. P. Csillag, Korlátos ingadozású függvények Fourier-féle állandóiról, *Math. és Phys. Lapok* **27**, 301–308, (1918).
18. L. Fejér, Über die Bestimmung des Sprunges der Funktion aus ihrer Fourierreihe, *J. Reine Angew. Math.* **142**, 165–188, (1913).
19. B.I. Golubov, Determination of the jumps of a function of bounded p-variation by its Fourier series, *Math. Notes* **12**, 444–449, (1972).
20. G. Kvernadze, Determination of the jumps of a bounded function by its Fourier series, *J. Approx. Theory* **92**, 167–190, (1998).
21. L.C. Young, Sur une généralisation de la notion de variation de puissance p-iéme bornée au sens de M. Wiener, et sur la convergence des séries de Fourier, *C. R. Acad. Sci. Paris* **204**, 470–472, (1937).
22. D. Waterman, On convergence of Fourier series of functions of generalized bounded variation, *Studia Math.* **44**, 107–117, (1972).
23. Z.A. Čanturija, The modulus of variation of a function and its application in the theory of Fourier series, *Dokl. Akad. Nauk* **15**, 67–71, (1974).
24. B.I. Golubov, Continuous functions of bounded p-variation, *Math. Notes* **1**, 203–207, (1967).
25. G. Kvernadze, Approximation of singularities of bounded function by partial sums of its differentiated Fourier series, to be submitted.